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Integral Means of Certain Analytic Functions for Fractional Calculus

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Abstract

Integral means inequalities with coefficients inequalities of certain analytic functions for the fractional derivatives and the fractional integral are determind by means of the subordination theorem. Relevant connections with known integral means with coefficients inequalities of analytic functions are also pointed out.

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1. Introduction

Let $\mathcal{A}_n$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N} = \{1, 2, 3, \cdots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $p(z)$ denote the analytic function in $\mathbb{U}$ defined by

$$p(z) = z + \sum_{s=1}^{m} b_{sj-s+1} z^{sj-s+1} \quad (j \geqq n+1; n \in \mathbb{N}). \tag{1.2}$$

In this paper, we shall discuss the integral means inequalities of $f(z)$ in $\mathcal{A}_n$ and $p(z)$ of the form (1.2) for the fractional derivative and the fractional integral.

In this chapter, we introduce our last work for the integral means inequalities. First, we need the concept of subordination for our investigation. For analytic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = g(w(z))$. We denote this subordination by $f(z) \prec g(z)$.

In 1925, Littlewood[2] proved the following subordination theorem, which is required for our investigation.

Theorem 1.1([2]). *If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $f(z) \prec g(z)$ $(z \in \mathbb{U})$, then for $\mu > 0$ and $z = re^{i\theta} (0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^\mu \, d\theta \leqq \int_0^{2\pi} |g(z)|^\mu \, d\theta.$$

Making use of Theorem 1.1, Silverman[3] proved the following theorem for analytic and univalent functions with negative coefficients.

Theorem 1.2([3]). *Let*

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geqq 0)$$

be analytic and univalent in $\mathbb{U}$. Then, for $z = re^{i\theta} (0 < r < 1)$ and $\mu > 0$,

$$\int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^\mu d\theta \leqq \int_0^{2\pi} \left| f_2\left(re^{i\theta}\right) \right|^\mu d\theta,$$

where $f_2(z) = z - z^2/2$.

We need the following Lemma.

Lemma 1.1([6]). *Let $P_m(t)$ denote the polynominal of degree $m (m \geqq 2)$ of the form*

$$P_m(t) = c_1 t^m - c_2 t^{m-1} - \cdots - c_{m-1} t^2 - c_m t - d \quad (t \geqq 0)$$

where $c_i (i = 1, 2, \cdots, m)$ are arbitrary positive constant and $d \geqq 0$. Then $P_m(t)$ has unique solution for $t > 0$. If we denote the solution by t_0, $P_m(t) < 0$ for $0 < t < t_0$ and $P_m(t) > 0$ for $t > t_0$.

Owa and Sekine[4] discussed the integral means with coefficients inequalities for the analytic functions $f(z)$ in $\mathcal{A}_n$ and $p(z)$ in the case where m=2 and 3.

Recently, Sekine, Owa and Yamakawa[6] proved the integral means inequalities of the analytic functions $f(z)$ in $\mathcal{A}_n$ and $p(z)(m \geqq 2)$. That is, applying Theorem 1.1 by Littlewood[2] and Lemma 1.1 of [6] above, we obtained the following results.

Theorem 1.3([6]). *Let the functions $f(z) \in \mathcal{A}_n$ and $p(z)(m \geqq 2)$ satisfy*

$$\sum_{k=n+1}^{\infty} |a_k| \leqq |b_{mj-m+1}| - \sum_{s=1}^{m-1} |b_{sj-s+1}|$$

with

$$|b_{mj-m+1}| > \sum_{s=1}^{m-1} |b_{sj-s+1}| .$$

If there exists an analytic function $w(z)$ in $\mathbb{U}$ defined by

$$\sum_{s=1}^{m} b_{sj-s+1}\{w(z)\}^{s(j-1)} - \sum_{k=n+1}^{\infty} a_k z^{k-1} = 0 ,$$

then for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^{\mu} d\theta \leqq \int_0^{2\pi} |p(z)|^{\mu} d\theta. \tag{1.3}$$

Further, by applying the Hölder inequality to the right hand side of the inequality (1.3) in Theorem 1.3, we proved the following integral mean inequality.

Corollary 1.1([6]). *If the functions $f(z) \in \mathcal{A}_n$ and $p(z)(m \geqq 2)$ satisfy the conditions in* Theorem 1.3, *then for $0 < \mu \leqq 2$ and $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\begin{aligned} \int_0^{2\pi} |f(z)|^{\mu} d\theta &\leqq 2\pi r^{\mu} \left(1 + \sum_{s=1}^{m} |b_{sj-s+1}|^2 r^{2s(j-1)}\right)^{\frac{\mu}{2}} \\ &< 2\pi \left(1 + \sum_{s=1}^{m} |b_{sj-s+1}|^2\right)^{\frac{\mu}{2}} . \end{aligned}$$

We obtained the integral means for the first derivative.

Theorem 1.4([6]). *Let the functions $f(z) \in \mathcal{A}_n$ and $p(z)(m \geqq 2)$ satisfy*

$$\sum_{k=n+1}^{\infty} k|a_k| \leqq (mj-m+1)|b_{mj-m+1}| - \sum_{s=1}^{m-1} (sj-s+1)|b_{sj-s+1}|$$

with

$$(mj-m+1)|b_{mj-m+1}| > \sum_{s=1}^{m-1} (sj-s+1)|b_{sj-s+1}|.$$

If there exists an analytic function $w(z)$ *in* $\mathbb{U}$ *defined by*

$$\sum_{s=1}^{m}(sj-s+1)b_{sj-s+1}\{w(z)\}^{s(j-1)}-\sum_{k=n+1}^{\infty}ka_kz^{k-1}=0,$$

then for $\mu>0$ *and* $z=re^{i\theta}$ $(0<r<1)$,

$$\int_0^{2\pi}|f'(z)|^{\mu}d\theta \leqq \int_0^{2\pi}|p'(z)|^{\mu}d\theta.$$

In the same way with Corollary 1.1, we obtained the integral mean inequality for $f'(z)$.

Corollary 1.2([6]). *If the functions* $f(z)\in\mathcal{A}_n$ *and* $p(z)(m\geqq 2)$ *satisfy the conditions in* Theorem 1.4, *then for* $0<\mu\leqq 2$ *and* $z=re^{i\theta}$ $(0<r<1)$,

$$\begin{aligned}\int_0^{2\pi}|f'(z)|^{\mu}d\theta &\leqq 2\pi r^{\mu}\left(1+\sum_{s=1}^{m}(sj-s+1)^2|b_{sj-s+1}|^2r^{2s(j-1)}\right)^{\frac{\mu}{2}}\\ &< 2\pi\left(1+\sum_{s=1}^{m}(sj-s+1)^2|b_{sj-s+1}|^2\right)^{\frac{\mu}{2}}.\end{aligned}$$

2. Integral Means for Fractional Calculus

We shall recall the following definitions of fractional calculus–that is, fractional integral and fractional derivative–by Owa[3] (see also Srivastava and Owa[9]).

Definition 2.1([3]). *The fractional integral of order* λ *is defined, for a function* $f(z)$, *by*

$$D_z^{-\lambda}f(z):=\frac{1}{\Gamma(\lambda)}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}\,d\zeta \quad (\lambda>0),$$

where the function $f(z)$ *is analytic in a simply-connected region of the complex* z-plane *containing the origin and the multiplicity of* $(z-\zeta)^{\lambda-1}$ *is removed by requiring* $\log(z-\zeta)$ *to be real when* $z-\zeta>0$.

Definition 2.2([3]). *The fractional derivative of order* λ *is defined, for a function* $f(z)$, *by*

$$D_z^{\lambda}f(z):=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{\lambda}}\,d\zeta \quad (0\leqq\lambda<1),$$

where the function $f(z)$ *is constrained, and the multiplicity of* $(z-\zeta)^{-\lambda}$ *is removed, as in* Definition 2.1 *above.*

Definition 2.3([3]). *Under the hypotheses of* Definition 2.2, *the fractional derivative of order $n+\lambda$ is defined, for a function $f(z)$, by*

$$D_z^{n+\lambda} f(z) := \frac{d^n}{dz^n} D_z^{\lambda} f(z) \quad (0 \leqq \lambda < 1;\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

By virtue of the Definitions 2.1, 2.2 and 2.3, we have

$$D_z^{-\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+\lambda+1)} z^{k+\lambda} \quad (k \in \mathbb{N},\ \lambda > 0), \tag{2.1}$$

$$D_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^{k-\lambda} \quad (k \in \mathbb{N},\ 0 \leqq \lambda < 1) \tag{2.2}$$

and

$$D_z^{q+\lambda} z^k = \frac{d^q}{dz^q} D_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k-q-\lambda+1)} z^{k-(q+\lambda)}$$
$$(q \in \mathbb{N}_0,\ k \in \mathbb{N},\ 0 \leqq \lambda < 1;\ q \leqq k \text{ for } \lambda = 0). \tag{2.3}$$

Applying the formulas of the fractional derivatives and fractional integral above, Kim and Choi[1], Sekine, Tsurumi and Srivastava[7], and Owa et al.[5] investigated some interesting properties for integral means of analytic functions for fractional calculus.

First, we have the following integral means for the fractional derivative.

Theorem 2.1 .

Let $f(z) \in A_n$ and $p(z)(m \geqq 2)$ be given by (1.1). *Suppose that*

$$\sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)}{|\Gamma(k+1-q-\lambda)|} |a_k|$$
$$\leqq \frac{|\Gamma(2-q-\nu)|}{|\Gamma(2-q-\lambda)|} \times \left\{ \frac{\Gamma(m(j-1)+2)}{|\Gamma(m(j-1)+2-q-\nu)|} |b_{m(j-1)+1}| - \sum_{s=1}^{m-1} \frac{\Gamma(s(j-1)+2)}{|\Gamma(s(j-1)+2-q-\nu)|} |b_{s(j-1)+1}| \right\}$$

with

$$\sum_{s=1}^{m-1} \frac{\Gamma(s(j-1)+2)}{|\Gamma(s(j-1)+2-q-\nu)|} |b_{s(j-1)+1}|$$
$$< \frac{\Gamma(m(j-1)+2)}{|\Gamma(m(j-1)+2-q-\nu)|} |b_{m(j-1)+1}|$$
$$(q \in \mathbb{N}_0,\ 0 \leqq \lambda,\ \nu < 1;\ q \leqq 1,\ \mathit{for}\ \lambda\nu = 0,\ j \geqq n+1;\ n \in \mathbb{N}). \tag{2.4}$$

If there exists an analytic function $w(z)$ in $\mathbb{U}$ defined by

$$\sum_{s=1}^{m} \frac{\Gamma(2-q-\nu)\Gamma(s(j-1)+2)}{\Gamma(s(j-1)+2-q-\nu)} b_{s(j-1)+1}\{w(z)\}^{s(j-1)} - \sum_{k=n+1}^{\infty} \frac{\Gamma(2-q-\lambda)\Gamma(k+1)}{\Gamma(k+1-q-\lambda)} a_k z^{k-1} = 0, \quad (2.5)$$

then for $z = re^{i\theta} (0 < r < 1)$ and $\mu > 0$,

$$\int_0^{2\pi} \left| D_z^{q+\lambda} f(z) \right|^{\mu} d\theta \leqq \left| \frac{\Gamma(2-q-\nu)}{\Gamma(2-q-\lambda)} \right| \int_0^{2\pi} \left| z^{\nu-\lambda} D_z^{q+\nu} p(z) \right|^{\mu} d\theta.$$

Proof. By means of the fractional derivative formula (2.3), we find from (1.1) that

$$D_z^{q+\lambda} f(z) = \frac{z^{1-q-\lambda}}{\Gamma(2-q-\lambda)} \left(1 + \sum_{k=n+1}^{\infty} \frac{\Gamma(2-q-\lambda)\Gamma(k+1)}{\Gamma(k+1-q-\lambda)} a_k z^{k-1} \right).$$

Also, by using the fractional derivative formula (2.3) for (1.2), we obtain

$$D_z^{q+\nu} p(z) = \frac{z^{1-q-\nu}}{\Gamma(2-q-\nu)} \left(1 + \sum_{s=1}^{m} \frac{\Gamma(2-q-\nu)\Gamma(s(j-1)+2)}{\Gamma(s(j-1)+2-q-\nu)} b_{s(j-1)+1} z^{s(j-1)} \right).$$

Thus we have

$$\begin{aligned} &\frac{\Gamma(2-q-\nu)}{\Gamma(2-q-\lambda)} z^{\nu-\lambda} D_z^{q+\nu} p(z) \\ &= \frac{z^{1-q-\lambda}}{\Gamma(2-q-\lambda)} \left(1 + \sum_{s=1}^{m} \frac{\Gamma(2-q-\nu)\Gamma(s(j-1)+1)}{\Gamma(s(j-1)+2-q-\nu)} b_{s(j-1)+1} z^{s(j-1)} \right). \end{aligned}$$

For $z = re^{i\theta}$ and $0 < r < 1$, we must show that

$$\begin{aligned} &\int_0^{2\pi} \left| 1 + \sum_{k=n+1}^{\infty} \frac{\Gamma(2-q-\lambda)\Gamma(k+1)}{\Gamma(k+1-q-\lambda)} a_k z^{k-1} \right|^{\mu} d\theta \\ &\leqq \int_0^{2\pi} \left| 1 + \sum_{s=1}^{m} \frac{\Gamma(2-q-\nu)\Gamma(s(j-1)+2)}{\Gamma(s(j-1)+2-q-\nu)} b_{s(j-1)+1} z^{s(j-1)} \right|^{\mu} d\theta \quad (\mu > 0). \end{aligned}$$

By applying Theorem 1.1, it would suffice to show that

$$\begin{aligned} &1 + \sum_{k=n+1}^{\infty} \frac{\Gamma(2-q-\lambda)\Gamma(k+1)}{\Gamma(k+1-q-\lambda)} a_k z^{k-1} \\ &\prec 1 + \sum_{s=1}^{m} \frac{\Gamma(2-q-\nu)\Gamma(s(j-1)+2)}{\Gamma(s(j-1)+2-q-\nu)} b_{s(j-1)+1} z^{s(j-1)}. \quad (2.6) \end{aligned}$$

Let us define the function $w(z)$ by

$$1+\sum_{k=n+1}^{\infty}\frac{\Gamma(2-q-\lambda)\Gamma(k+1)}{\Gamma(k+1-q-\lambda)}a_k z^{k-1}$$
$$=1+\sum_{s=1}^{m}\frac{\Gamma(2-q-\nu)\Gamma(s(j-1)+2)}{\Gamma(s(j-1)+2-q-\nu)}b_{s(j-1)+1}\{w(z)\}^{s(j-1)}. \quad (2.7)$$

Thus, it follows that

$$\{w(0)\}^{j-1}\sum_{s=1}^{m}\frac{\Gamma(2-q-\nu)\Gamma(s(j-1)+2)}{\Gamma(s(j-1)+2-q-\nu)}b_{s(j-1)+1}\{w(0)\}^{(s-1)(j-1)}=0.$$

Therefore, if there exists an analytic functions $w(z)$ which satisfies the equality (2.5), we have an analytic function $w(z)$ in $\mathbb{U}$ such that $w(0)=0$.

Further, we prove that the analytic function $w(z)$ satisfies $|w(z)|<1 (z\in\mathbb{U})$ for (2.5). From the equality (2.7), we know that

$$\left|\sum_{s=1}^{m}\frac{\Gamma(2-q-\nu)\Gamma(s(j-1)+2)}{\Gamma(s(j-1)+2-q-\nu)}b_{s(j-1)+1}\{w(z)\}^{s(j-1)}\right|$$
$$\leqq\sum_{k=n+1}^{\infty}\frac{|\Gamma(2-q-\lambda)|\Gamma(k+1)}{|\Gamma(k+1-q-\lambda)|}|a_k z^{k-1}|$$
$$<\sum_{k=n+1}^{\infty}\frac{|\Gamma(2-q-\lambda)|\Gamma(k+1)}{|\Gamma(k+1-q-\lambda)|}|a_k| \quad (2.8)$$

for $z\in\mathbb{U}$, so that

$$\frac{|\Gamma(2-q-\nu)|\Gamma(m(j-1)+2)}{|\Gamma(m(j-1)+2-q-\nu)|}|b_{m(j-1)+1}|\left|\{w(z)\}^{m(j-1)}\right|$$
$$-\left|\sum_{s=1}^{m-1}\frac{\Gamma(2-q-\nu)\Gamma(s(j-1)+2)}{\Gamma(s(j-1)+2-q-\nu)}b_{s(j-1)+1}\{w(z)\}^{s(j-1)}\right|$$
$$-\sum_{k=n+1}^{\infty}\frac{|\Gamma(2-q-\lambda)|\Gamma(k+1)}{|\Gamma(k+1-q-\lambda)|}|a_k|<0$$

for $z\in\mathbb{U}$.

Putting $t=|w(z)|^{j-1}$ $(t\geqq 0)$, we define the polynomial $Q(t)$ of degree m by

$$Q(t)=\frac{|\Gamma(2-q-\nu)|\Gamma(m(j-1)+2)}{|\Gamma(m(j-1)+2-q-\nu)|}|b_{m(j-1)+1}|t^m$$
$$-\sum_{s=1}^{m-1}\frac{|\Gamma(2-q-\nu)|\Gamma(s(j-1)+2)}{|\Gamma(s(j-1)+2-q-\nu)|}|b_{s(j-1)+1}|t^s$$
$$-\sum_{k=n+1}^{\infty}\frac{|\Gamma(2-q-\lambda)|\Gamma(k+1)}{|\Gamma(k+1-q-\lambda)|}|a_k|$$

By means of Lemma 1.1, if $Q(1) \geqq 0$, we have $t < 1$ for $Q(t) < 0$. Hence for $|w(z)| < 1$ $(z \in \mathbb{U})$, we need the following inequality

$$Q(1) = \frac{|\Gamma(2-q-\nu)|\Gamma(m(j-1)+2)}{|\Gamma(m(j-1)+2-q-\nu)|}|b_{m(j-1)+1}| - \sum_{s=1}^{m-1} \frac{|\Gamma(2-q-\nu)|\Gamma(s(j-1)+2)}{|\Gamma(s(j-1)+2-q-\nu)|}|b_{s(j-1)+1}| - \sum_{k=n+1}^{\infty} \frac{|\Gamma(2-q-\lambda)|\Gamma(k+1)}{|\Gamma(k+1-q-\lambda)|}|a_k| \geqq 0,$$

that is,

$$\sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)}{|\Gamma(k+1-q-\lambda)|}|a_k| \leqq \frac{|\Gamma(2-q-\nu)|}{|\Gamma(2-q-\lambda)|} \times \left\{ \frac{\Gamma(m(j-1)+2)}{|\Gamma(m(j-1)+2-q-\nu)|}|b_{m(j-1)+1}| - \sum_{s=1}^{m-1} \frac{\Gamma(s(j-1)+2)}{|\Gamma(s(j-1)+2-q-\nu)|}|b_{s(j-1)+1}| \right\}$$

Therefore the subordination in (2.6) holds true, and this evidently completes the proof of Theorem 2.1.

In case of $m = 1$, see Owa et al.[5], and Owa and Skine[4] for $m = 2$ and 3.

Remark 2.1.
If $q = 0$ and $\lambda = \nu = 0$ in Theorem 2.1, we have Theorem 1.3. Also, when $q = 1$ and $\lambda = \nu = 0$ in Theorem 2.1, Theorem 2.1 coincides with Theorem 1.4.

Putting $\nu = \lambda$ in Theorem 2.1, we have the integral means for the fractional derivative of order $q + \lambda$.

Corollary 2.1. *Let $f(z) \in A_n$ and $p(z) (m \geqq 2)$ be given by* (1.1). *Supposed that*

$$\sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)}{|\Gamma(k+1-q-\lambda)|}|a_k| \leqq \frac{\Gamma(m(j-1)+2)}{|\Gamma(m(j-1)+2-q-\lambda)|}|b_{m(j-1)+1}| - \sum_{s=1}^{m-1} \frac{\Gamma(s(j-1)+2)}{|\Gamma(s(j-1)+2-q-\lambda)|}|b_{s(j-1)+1}|$$

with

$$\sum_{s=1}^{m-1} \frac{\Gamma(s(j-1)+2)}{|\Gamma(s(j-1)+2-q-\lambda)|}|b_{s(j-1)+1}| < \frac{\Gamma(m(j-1)+2)}{|\Gamma(m(j-1)+2-q-\lambda)|}|b_{m(j-1)+1}|$$

$$(q \in \mathbb{N}_0,\ 0 \leqq \lambda < 1;\ q \leqq n{+}1 \ for\ \lambda = 0,\ j \geqq n{+}1;\ n \in \mathbb{N}).$$

If there exists an analytic function $w(z)$ in $\mathbb{U}$ defined by

$$\sum_{s=1}^{m} \frac{\Gamma(s(j-1)+2)}{\Gamma(s(j-1)+2-q-\lambda)} b_{s(j-1)+1}\{w(z)\}^{s(j-1)} - \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1-q-\lambda)} a_k z^{k-1} = 0,$$

then for $z = re^{i\theta}(0 < r < 1)$ and $\mu > 0$,

$$\int_0^{2\pi} \left| D_z^{q+\lambda} f(z) \right|^{\mu} d\theta \leqq \int_0^{2\pi} \left| D_z^{q+\lambda} p(z) \right|^{\mu} d\theta. \tag{2.9}$$

Applying the Hölder inequality to the right hand side of the inequality (2.9) in Corollary 2.1, we obtain the following integral mean inequality.

Corollary 2.2. *If the functions $f(z) \in \mathcal{A}_n$ and $p(z)(m \geqq 2)$ satisfy the conditions in* Corollary 2.1, *then for $0 < \mu \leqq 2$ and* $z = re^{i\theta}\,(0 < r < 1)$,

$$\begin{aligned}
&\int_0^{2\pi} \left| D_z^{q+\lambda} f(z) \right|^{\mu} d\theta \\
&\leqq \frac{2\pi r^{(1-q-\lambda)\mu}}{|\Gamma(2-q-\lambda)|^{\mu}} \left(1 + \sum_{s=1}^{m} \frac{|\Gamma(2-q-\lambda)|\Gamma(s(j-1)+2)}{|\Gamma(s(j-1)+2-q-\lambda)|} |b_{sj-s+1}|^2 r^{2s(j-1)} \right)^{\frac{\mu}{2}} \\
&< \frac{2\pi}{|\Gamma(2-q-\lambda)|^{\mu}} \left(1 + \sum_{s=1}^{m} \frac{|\Gamma(2-q-\lambda)|\Gamma(s(j-1)+2)}{|\Gamma(s(j-1)+2-q-\lambda)|} |b_{sj-s+1}|^2 \right)^{\frac{\mu}{2}}
\end{aligned}$$

$$(q \in \mathbb{N}_0,\ 0 \leqq \lambda < 1;\ q \leqq 1 \ for\ \lambda = 0;\ j \geqq n+1,\ n \in \mathbb{N}).$$

Proof. Since,

$$\begin{aligned}
&\int_0^{2\pi} |D_z^{q+\lambda} p(z)|^{\mu} d\theta \\
&= \int_0^{2\pi} \left| \frac{z^{1-q-\lambda}}{\Gamma(2-q-\lambda)} \right|^{\mu} \left| 1 + \sum_{s=1}^{m} \frac{\Gamma(2-q-\lambda)\Gamma(s(j-1)+2)}{\Gamma(s(j-1)+2-q-\lambda)} b_{s(j-1)+1} z^{s(j-1)} \right|^{\mu} d\theta.
\end{aligned}$$

Making use of the inequality of Hölder for $0 < \mu < 2$, we obtain that

$$\begin{aligned}
\int_0^{2\pi} |D_z^{q+\lambda} p(z)|^{\mu} d\theta &\leqq \left(\int_0^{2\pi} \left(\left| \frac{z^{1-q-\lambda}}{\Gamma(2-q-\lambda)} \right|^{\mu} \right)^{\frac{2}{2-\mu}} d\theta \right)^{\frac{2-\mu}{2}} \\
&\times \left\{ \int_0^{2\pi} \left(\left| 1 + \sum_{s=1}^{m} \frac{\Gamma(2-q-\lambda)\Gamma(s(j-1)+2)}{\Gamma(s(j-1)+2-q-\lambda)} b_{s(j-1)+1} z^{s(j-1)} \right|^{\mu} \right)^{\frac{2}{\mu}} d\theta \right\}^{\frac{\mu}{2}}
\end{aligned}$$

$$= \left(\frac{r^{\frac{(1-q-\lambda)2\mu}{2-\mu}}}{|\Gamma(2-q-\lambda)|^{\frac{2\mu}{2-\mu}}} \int_0^{2\pi} d\theta \right)^{\frac{2-\mu}{2}}$$

$$\times \left(\int_0^{2\pi} \left| 1 + \sum_{s=1}^{m} \frac{\Gamma(2-q-\lambda)\Gamma(s(j-1)+2)}{\Gamma(s(j-1)+2-q-\lambda)} b_{s(j-1)+1} z^{s(j-1)} \right|^2 d\theta \right)^{\frac{\mu}{2}}$$

$$= \left(\frac{2\pi r^{\frac{(1-q-\lambda)2\mu}{2-\mu}}}{|\Gamma(2-q-\lambda)|^{\frac{2\mu}{2-\mu}}} \right)^{\frac{2-\mu}{2}}$$

$$\times \left\{ 2\pi \left(1 + \sum_{s=1}^{m} \frac{|\Gamma(2-q-\lambda)|\Gamma(s(j-1)+2)}{|\Gamma(s(j-1)+2-q-\lambda)|} |b_{s(j-1)+1}|^2 r^{2s(j-1)} \right) \right\}^{\frac{\mu}{2}}$$

$$= \frac{2\pi r^{(1-q-\lambda)\mu}}{|\Gamma(2-q-\lambda)|^{\mu}} \left(1 + \sum_{s=1}^{m} \frac{|\Gamma(2-q-\lambda)|\Gamma(s(j-1)+2)}{|\Gamma(s(j-1)+2-q-\lambda)|} |b_{s(j-1)+1}|^2 r^{2s(j-1)} \right)^{\frac{\mu}{2}}$$

$$< \frac{2\pi}{|\Gamma(2-q-\lambda)|^{\mu}} \left(1 + \sum_{s=1}^{m} \frac{|\Gamma(2-q-\lambda)|\Gamma(s(j-1)+2)}{|\Gamma(s(j-1)+2-q-\lambda)|} |b_{s(j-1)+1}|^2 \right)^{\frac{\mu}{2}}.$$

It is easy to show the case of $\mu = 2$. This comples the proof of Corollary 2.2.

Remark 2.2.

If we put $q = 0$ and $\lambda = 0$ in Corollary 2.2, we have Corollary 1.1. Also, when $q = 1$ and $\lambda = 0$ in Corollary 2.2, Corollary 2.2 coincides with Corollary 1.2.

Lastly, by means of the fractional formulas (2.1), (2.2) and (2.3), replacing λ by $-\lambda(\lambda > 0)$, ν by $-\nu(\nu > 0)$, and q by $-q(q \in \mathbb{N}_0)$ in Theorem 2.1, we have the following integral means inequality for the fractional integral.

Theorem 2.2. *Let $f(z) \in A_n$ and $p(z)(m \geqq 2)$ be given by (1.1). Supposed that*

$$\sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1+q+\lambda)} |a_k|$$
$$\leqq \frac{\Gamma(2+q+\nu)}{\Gamma(2+q+\lambda)} \times \left\{ \frac{\Gamma(m(j-1)+2)}{\Gamma(m(j-1)+2+q+\nu)} |b_{m(j-1)+1}| - \sum_{s=1}^{m-1} \frac{\Gamma(s(j-1)+2)}{\Gamma(s(j-1)+2+q+\nu)} |b_{s(j-1)+1}| \right\}$$

with

$$\sum_{s=1}^{m-1} \frac{\Gamma(s(j-1)+2)}{\Gamma(s(j-1)+2+q+\nu)} |b_{s(j-1)+1}| < \frac{\Gamma(m(j-1)+2)}{\Gamma(m(j-1)+2+q+\nu)} |b_{m(j-1)+1}|$$
$$(q \in \mathbb{N}_0,\ 0 < \lambda,\ \nu < 1).$$

If there exists an analytic function $w(z)$ in $\mathbb{U}$ defined by

$$\sum_{s=1}^{m}\frac{\Gamma(2+q+\nu)\Gamma(s(j-1)+2)}{\Gamma(s(j-1)+2+q+\nu)}b_{s(j-1)+1}\{w(z)\}^{s(j-1)} - \sum_{k=n+1}^{\infty}\frac{\Gamma(2+q+\lambda)\Gamma(k+1)}{\Gamma(k+1+q+\lambda)}a_k z^{k-1}=0,$$

then for $z=re^{i\theta}(0<r<1)$ and $\mu>0$,

$$\int_0^{2\pi}\left|D_z^{-(q+\lambda)}f(z)\right|^{\mu}d\theta \leqq \left|\frac{\Gamma(2+q+\nu)}{\Gamma(2+q+\lambda)}\right|\int_0^{2\pi}|z^{-\nu+\lambda}D_z^{-(q+\nu)}f(z)|^{\mu}d\theta.$$

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